Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number | Candidate Number

Pearson Edexcel Level 3 GCE

Monday 5 June 2023

Afternoon (Time: 1 hour 30 minutes)

Paper reference **9FM0/02**

Further Mathematics

Advanced

PAPER 2: Core Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

1.

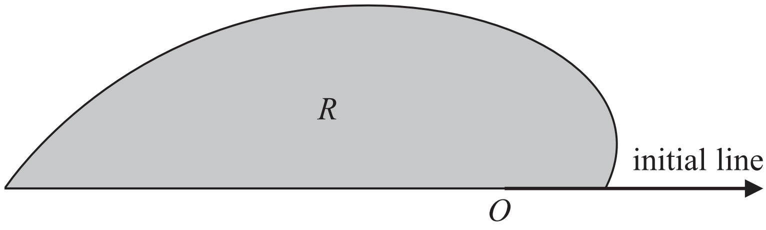


Figure 1

Figure 1 shows a sketch of the curve with polar equation

$$r = 2\sqrt{\sinh\theta + \cosh\theta} \qquad 0 \leqslant \theta \leqslant \pi$$

The region R, shown shaded in Figure 1, is bounded by the initial line, the curve and the line with equation $\theta = \pi$

Use algebraic integration to determine the exact area of R giving your answer in the form $pe^q - r$ where p, q and r are real numbers to be found.

(4)

remembering the formula for polar integration : $\frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$

and subbing in what we have in the question

$\alpha = 0, \beta = \pi$

$r = 2\sqrt{\sinh\theta + \cosh\theta}$

$r^2 = 4(\sinh\theta + \cosh\theta)$

taking the 4 out to front of polar integration

$\frac{1}{2}(4)\int_0^{\pi} \sinh\theta + \cosh\theta \, d\theta$

↳ notice this involves the integration of HYPERBOLIC FUNCTIONS

WAY 1: using $\int \sinh x \, dx = \cosh x + c$

$\int \cosh x \, dx = \sinh x + c$

$2[\cosh\theta + \sinh\theta]_0^{\pi}$

↳ evaluate above in limits

$2\{(\cosh(\pi) + \sinh(\pi)) - \cosh(0) - \sinh(0))\}$

want exact area so not evaluating using calculator but using exponential definitions of $\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$2\{\frac{1}{2}(e^{\pi} + e^{-\pi}) + \frac{1}{2}(e^{\pi} - e^{-\pi}) - (1) - (0)\}$

$\Rightarrow \quad 2\left(\frac{\cancel{2}e^{\pi}}{\cancel{2}} - 1\right) = \boxed{2e^{\pi} - 2} \qquad \Rightarrow \boxed{p = 2, q = \pi, r = 2}$

Question 1 continued

WAY 2 : integrating exponential DFN : $\cosh\theta = \frac{1}{2}(e^{\theta}+e^{-\theta})$

$\sinh\theta = \frac{1}{2}(e^{\theta}-e^{-\theta})$

$$2\int_0^{\pi}\left(\frac{e^{\theta}-\cancel{e^{-\theta}}}{2}+\frac{e^{\theta}+\cancel{e^{-\theta}}}{2}\right)$$

$$2\int_0^{\pi}\left(\frac{\cancel{2}e^{\theta}}{\cancel{2}}\right)d\theta$$

$$2\int_0^{\pi}e^{\theta}\,d\theta$$

$\Rightarrow 2\left[e^{\theta}\right]_0^{\pi}$

$\Rightarrow 2\{e^{\pi}-e^{0}\}$

$\Rightarrow 2(e^{\pi}-1)$

$= \boxed{2e^{\pi}-2}$ where $\boxed{p=2,\ q=\pi \text{ and } r=2}$

(Total for Question 1 is 4 marks)

Year 2 Series - using compound substitution into formula book Maclaurin series

2. (a) Write down the Maclaurin series of e^x, in ascending power of x, up to and including the term in x^3

(1)

(b) Hence, without differentiating, determine the Maclaurin series of

$$e^{(e^x - 1)}$$

in ascending powers of x, up to and including the term in x^3, giving each coefficient in simplest form.

(5)

(a) from FORMULA BOOKLET :

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

(b) knowing there are two ways to determine the Maclaurin series of certain expressions : one is through differentiating expression and evaluating its derivatives at 0 (filling in general Maclaurin series formula) OR using compound function substitution into already existing Maclaurin series formulae

∴ here need to do the latter :

WAY 1 : substitution of e^{x-1} into formula book expansion

first consider series expansion : $e^x - 1$

$$= \underbrace{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)}_{\text{part (a)}} - 1$$

$$= x + \frac{x^2}{2} + \frac{x^3}{6}$$

now substituting $x \to x + \frac{x^2}{2} + \frac{x^3}{6}$ into the formula book expansion of e^x

$$e^{e^x - 1} = 1 + \left(x + \frac{x^2}{2} + \frac{x^3}{6}\right) + \frac{\left(x + \frac{x^2}{2} + \frac{x^3}{6}\right)^2}{2!} + \frac{\left(x + \frac{x^2}{2} + \frac{x^3}{6}\right)^3}{3!} + \ldots$$

$$= 1 + \left(x + \frac{x^2}{2} + \frac{x^3}{6}\right) + \frac{\left(x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(x + \frac{x^2}{2} + \frac{x^3}{6}\right)}{2!} + \frac{\left(x + \frac{x^2}{2} + \frac{x^3}{6}\right)^3}{3!}$$

expanding above but only up to and including x^3

eg. $\left(x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(x + \frac{x^2}{2} + \frac{x^3}{6}\right)$

becomes

$$= x^2 + \frac{x^3}{2} + \frac{x^3}{2} + \ldots$$

$$= x^2 + x^3$$

P 7 2 7 9 5 A 0 4 3 2

Question 2 continued

and $\left(x+\frac{x^2}{2}+\frac{x^3}{6}\right)^3$ become x^3

$\therefore e^{e^x-1} = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^2+x^3}{2!}+\frac{x^3}{6}$

$= 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^3}{6}$

collect like terms:

$\boxed{= 1+x+x^2+\frac{5x^3}{6}+\ldots}$

WAY 2: product of function (using index laws)

consider $e^x-1 = \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)-1$

$= x+\frac{x^2}{2}+\frac{x^3}{6}$

substituting into e^x formula booklet

↳ where $x \to x+\frac{x^2}{2}+\frac{x^3}{6}$

$e^{x+\frac{x^2}{2}+\frac{x^3}{6}}$ "up to and including x^3"

using index power rule i.e splitting above into product of 3 separate functions

$e^x \times e^{x^2/2} \times e^{x^3/6}$

evaluating each separately (up to and including x^3)

$e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{6}$

$e^{x^2/2} = 1+\left(\frac{x^2}{2}\right)+\cancel{\left(\frac{\left(\frac{x^2}{2}\right)^2}{2}\right)}$

$= 1+\frac{x^2}{2}$

$e^{x^3/6} = 1+\frac{x^3}{6}$

and multiplying

$\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)\left(1+\frac{x^2}{2}\right)\left(1+\frac{x^3}{6}\right)$

expand first two brackets (up to and including x^3)

$\left(1+\frac{x^2}{2}+x+\frac{x^3}{2}+\frac{x^2}{2}+\cancel{\frac{x^4}{4}}+\frac{x^3}{6}+\cancel{\frac{x^5}{12}}\right)\left(1+\frac{x^3}{6}\right)$

collect like terms

$\left(1+x+x^2+\frac{2}{3}x^3\right)\left(1+\frac{x^3}{6}\right)$

expand out

$1+x+x^2+\frac{2}{3}x^3+\frac{x^3}{6}$

$\boxed{= 1+x+x^2+\frac{5}{6}x^3+\ldots}$

(Total for Question 2 is 6 marks)

3.

$$\mathbf{M} = \begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix}$$

where k is a constant.

Given that

$$\mathbf{M}^2 + 11\mathbf{M} = a\mathbf{I}$$

where a is a constant and $\mathbf{I}$ is the 2×2 identity matrix,

(a) (i) determine the value of a

(ii) show that $k = -9$

(3)

(b) Determine the equations of the invariant lines of the transformation represented by $\mathbf{M}$.

(6)

(c) State which, if any, of the lines identified in (b) consist of fixed points, giving a reason for your answer.

(1)

(a)(i) first evaluate LHS to above equation, so:

$M^2 + 11M$

...for M^2: matrix multiplication "rows into columns"

$$\begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix}\begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix}$$

multiply the elements in the row of the first matrix by the elements in the column of the second matrix and sum in between

$$\begin{pmatrix} \begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix}\begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix} & \begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix}\begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix} \\ \begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix}\begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix} & \begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix}\begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} (-2)(-2) + 5(6) & (-2)(5) + (5)k \\ 6(-2) + k(6) & 6(5) + k(k) \end{pmatrix}$$

$$= \begin{pmatrix} 34 & 5k-10 \\ 6k-12 & k^2+30 \end{pmatrix}$$

for $11M$: just scalar multiplication (multiply elements by 11)

$$11M = \begin{pmatrix} -2(11) & 5(11) \\ 6(11) & 11k \end{pmatrix}$$

$$= \begin{pmatrix} -22 & 55 \\ 66 & 11k \end{pmatrix}$$

Question 3 continued

L.H.S

$M^2+11M = \begin{pmatrix} 34 & 5k-10 \\ 6k-12 & k^2+30 \end{pmatrix} + \begin{pmatrix} -22 & 55 \\ 66 & 11k \end{pmatrix}$

matrix addition (add corresponding elements)

$\begin{pmatrix} 34-22 & 5k-10+55 \\ 6k-12+66 & k^2+30+11k \end{pmatrix} = \begin{pmatrix} 12 & 5k+45 \\ 54+6k & k^2+11k+30 \end{pmatrix}$

now RHS - remembering that a 2x2 identity matrix is just a 2x2 zero matrix with '1's in the main diagonal (multiplied by 'a':

$RHS = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

$\therefore$ LHS = RHS

$\begin{pmatrix} 12 & 5k+45 \\ 54+6k & k^2+11k+30 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

can equate corresponding elements:

... A(1,1):

$\boxed{a = 12}$

(ii) ... A(1,2):

$5k+45 = 0$

$5k = -45$ $\div 5$

$\div 5$ $\boxed{k = -9}$

OR A(2,1)

$54+6k = 0$

$\Rightarrow 6k = -54$ $\div 6$

$\div 6$ $\boxed{k = -9}$

OR $k^2+11k+30 = "12"$

$\Rightarrow k^2+11k+18 = 0$

$\Rightarrow k = -2$ or -9

consider A(1,2)

$54+6(-2) \neq 0$

$\boxed{\therefore k = -9}$

(b) METHOD 1: transforming line $y = mx+c$

an invariant line is a line of points (call it $y=mx+c$ $\therefore$ as a vector $\begin{pmatrix} x \\ mx+c \end{pmatrix}$) which under the transformation M would be mapped to a different point i.e $\begin{pmatrix} x' \\ mx'+c \end{pmatrix}$ on the SAME straight line

formulating this as an equation (using $Mx = y$)

$\begin{pmatrix} -2 & 5 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} x' \\ mx'+c \end{pmatrix}$

matrix multiplication "rows into columns": product matrix $\begin{pmatrix} A(1,1) \\ A(2,1) \end{pmatrix}$

...for A(1,1):

$-2(x) + 5(mx+c)$

$= -2x + 5mx + 5c$

Question 3 continued

$\therefore$ for $A(2,1)$:

$6x + (-9)(mx+c) = mx' + c$

expand

$6x - 9mx - 9c = mx' + c$

equating to R.H.S:

$\begin{pmatrix} -2x+5mx+5c \\ 6x-9mx-9c \end{pmatrix} = \begin{pmatrix} x' \\ mx'+c \end{pmatrix}$

$\Rightarrow -2x + 5mx + 5c = x' \quad -①$

$6x - 9mx - 9c = mx' + c \quad -②$

sub ① into ②

$6x - 9mx - 9c = m(-2x + 5mx + 5c) + c$

expand

$6x - 9mx - 9c = -2mx + 5m^2x + 5mc + c$

collect x's and c's on either side

$-5mc - 10c = 5m^2x + 7mx - 6x$

$\Rightarrow x(5m^2 + 7m - 6) = -5c(m+2)$

$\Rightarrow x(5m^2 + 7m - 6) + 5c(m+2) = 0$

now make each bracket equal 0

$5m^2 + 7m - 6 = 0$

$4ac = 30, b = 7$

$10, -3$

$5m^2 + 10m - 3m - 6 = 0$

$\Rightarrow 5m(m+2) - 3(m+2) = 0$

$\Rightarrow (5m-3)(m+2) = 0$

$\Rightarrow m = -2, 3/5$

when $m = -2$,

$5c(-2+2) = 0$

$\therefore$ 'c' can take any value

subbing back into original

$\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} x \\ -2x+c \end{pmatrix}$

$\boxed{\therefore y = -2x + c}$

and when $n = 3/5$,

$5c(-3/5 + 2) \neq 0$

$\begin{pmatrix} x \\ 3/5x \end{pmatrix} \quad \therefore c = 0$

$\boxed{\therefore y = 3/5x}$

METHOD 2: transformation of points

an invariant line is a line of points - let the points be $\begin{pmatrix} x \\ y \end{pmatrix}$, each of which under the transformation are mapped to another point $\begin{pmatrix} x' \\ y' \end{pmatrix}$ on the same line: $y = mx + c$

↳ formulating this as an equation

$$\begin{pmatrix} -2 & 5 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

matrix multiplication "rows into columns"

$$\begin{pmatrix} -2x + 5y \\ 6x - 9y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

equate to R.H.S and form linear equations:

$$-2x + 5y = x'$$
$$6x - 9y = y'$$

now subbing in $y = mx + c$ in both so lie on same straight line

$$-2x + 5(mx + c) = x'$$
$$-2x + 5mx + 5c = x'$$

...factorise x's and c's:

$$x(5m - 2) + 5c = x'$$

$$6x - 9(mx + c) = y'$$
$$6x - 9mx - 9c = y'$$

factorise x's and c's:

$$x(6 - 9m) - 9c = y'$$

Sub into formula for transformed points: $y' = mx' + c$

$$x(6 - 9m) - 9c = m\left[x(5m - 2) + 5c\right] + c$$

expand square brackets:

$$x(6 - 9m) - 9c = x(5m^2 - 2m) + c + 5mc$$

compare coefficients

...x:

$$6 - 9m = 5m^2 - 2m$$
$$\Rightarrow 5m^2 + 7m - 6 = 0$$

need to solve

$ac = -30$

$b = 7$

$-3, 10$

$$5m^2 + 10m - 3m - 6 = 0$$
$$5m(m + 2) - 3(m + 2) = 0$$
$$\Rightarrow (5m - 3)(m + 2) = 0$$
$$\Rightarrow m = -2 \text{ or } 3/5$$

when $m = -2$,

$$\therefore c(5m + 10)$$
$$= c(5(-2) + 10))$$
$$\Rightarrow c = 0$$

$\therefore$ subbing into origin

$$\boxed{y = -2x + c}$$

when $m = 3/5$

$$c(5(3/5) + 10) \neq 0$$

$\therefore c$ has to be 0

$\therefore$ Subbing into original: $\boxed{y = 3/5\,x}$

Question 3 continued

(c) if 'fixed points' mean any point on invariant lines in (b) get mapped onto themselves

...try: $y = 3/5x$

taking a random point $(a, 3/5 a)$ and 'transform' to see if get $\begin{pmatrix} a \\ 3/5 a \end{pmatrix}$

$$\begin{pmatrix} -2 & 5 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} a \\ 3/5 a \end{pmatrix}$$

matrix multiplication - "rows into columns" let product matrix $= \begin{pmatrix} A_{(1,1)} \\ A_{(2,1)} \end{pmatrix}$

...for $A_{(1,1)}$:

$-2a + \cancel{5}(3/\cancel{5} a)$

$= a$ (✓ AIM)

...for $A_{(1,2)}$:

$6(a) - 9(3/5 a)$

$= 6a - \frac{27}{5}a$

$= 3/5 a$ (✓ AIM)

$\therefore y = 3/5 x$

trying $y = -2x + c$

random points: $\begin{pmatrix} a \\ -2a + c \end{pmatrix}$

see if get $\begin{pmatrix} a \\ -2a + c \end{pmatrix}$ mapped

$$\begin{pmatrix} -2 & 5 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} a \\ -2a + c \end{pmatrix}$$

matrix multiplication - "rows into columns"

product matrix $= \begin{pmatrix} A_{(1,1)} \\ A_{(2,1)} \end{pmatrix}$

... for $A_{(1,1)}$:

$-2a + (5)(-2a) + 5c$

$= -2a - 10a + 5c$

$= -12a + 5c \neq a$

$\therefore$ not fixed points

$\boxed{\therefore \text{ only } y = 3/5 x \text{ consistents of fixed points}}$

(Total for Question 3 is 10 marks)

4. (a) Sketch the polar curve C, with equation

$$r = 3 + \sqrt{5}\cos\theta \qquad 0 \leqslant \theta \leqslant 2\pi$$

On your sketch clearly label the pole, the initial line and the value of r at the point where the curve intersects the initial line.

(2)

The tangent to C at the point A, where $0 < \theta < \frac{\pi}{2}$, is parallel to the initial line.

(b) Use calculus to show that at A

$$\cos\theta = \frac{1}{\sqrt{5}}$$

(4)

(c) Hence determine the value of r at A.

(1)

(a) equation given in the form $a(p+q\cos\theta)$ – remembering the different cases for sketching polar graphs:

CASE 1: when $p=q$, get a cardioid (dimple at origin)

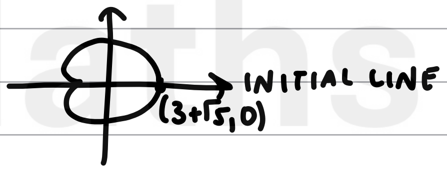
CASE 2: when $p \geq 2q$, get an oval / 'egg'

CASE 3: when $q<p<2q$, get a dimple (centre NOT at origin)

now need table of values to see where dimple located – go up in increments of $\pi/2$ in RANGE $0 \leq \theta \leq 2\pi$

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	$3+\sqrt{5}(\cos(0))$ $=3+\sqrt{5}$	3	$3+\sqrt{5}(-1)$ $3-\sqrt{5}$	$3+\sqrt{5}(0)$ $=3$	$3+\sqrt{5}(1)$ $=3+\sqrt{5}$

← dimple



(b) we know that for a tangent to be parallel to initial line, $\frac{dy}{d\theta}=0$ (this is because gradient of horizontal lines is 0, so for the parametric differentiation of hyperbolic functions) :

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

∴ numerator has to be equal to 0

∴ using parametric definition of (x,y) i.e $(r\cos\theta, r\sin\theta)$

$$y = r\sin\theta$$

sub in our 'r'

Question 4 continued

$y=(3+\sqrt{5}\cos\theta)\sin\theta$

now need to differentiate above:

WAY 1: product rule

$u=3+\sqrt{5}\cos\theta \quad v=\sin\theta$

$u'=-\sqrt{5}\sin\theta \quad v'=\cos\theta$

$\frac{d}{d\theta}(uv)=uv'+u'v$

$\Rightarrow \frac{dy}{d\theta}=(3+\sqrt{5}\cos\theta)\cos\theta + (-\sqrt{5}\sin\theta)\sin\theta$

$=3\cos\theta+\sqrt{5}\cos^2\theta-\sqrt{5}\sin^2\theta$

and solve $\frac{dy}{d\theta}=0$

convert $\sin^2\theta$ to $1-\cos^2\theta$ (using Pythag. identity)

$3\cos\theta+\sqrt{5}\cos^2\theta-\sqrt{5}(1-\cos^2\theta)=0$

expand

$2\sqrt{5}\cos^2\theta+3\cos\theta-\sqrt{5}=0$

WAY 2: using sin double angle

expand brackets:

$y=3\sin\theta+\sqrt{5}\sin\theta\cos\theta$

using $\sin\theta\cos\theta=\frac{1}{2}\sin 2\theta$

$\Rightarrow (\sqrt{5}\sin\theta\cos\theta=\frac{\sqrt{5}}{2}\sin 2\theta)$

$\therefore y=3\sin\theta+\frac{\sqrt{5}}{2}\sin 2\theta \Rightarrow \frac{dy}{d\theta}=3\cos\theta+\frac{2\sqrt{5}}{2}\cos 2\theta$

using memorised rearranged $\cos 2\theta = 2\cos^2\theta-1$

$0=3\cos\theta+\sqrt{5}(2\cos^2\theta-1)$

$=3\cos\theta+2\sqrt{5}\cos^2\theta-\sqrt{5}=0$

$\Rightarrow 2\sqrt{5}\cos^2\theta+3\cos\theta-\sqrt{5}=0$

$\therefore$ solving this quadratic in $\cos\theta$ – using quadratic formula / equation solver

$\cos\theta=\frac{-3\pm\sqrt{(3)^2-4(2\sqrt{5})(-\sqrt{5})}}{2(2\sqrt{5})}$

$=\frac{-3\pm\sqrt{9+40}}{4\sqrt{5}}$

$=\frac{-3\pm 7}{4\sqrt{5}}$

$\therefore \cos\theta=\frac{-\cancel{10}^{5}}{\cancel{4}_{2}\sqrt{5}} < 1 \therefore$ no real solutions

$\boxed{\therefore \cos\theta=\frac{\cancel{4}}{\cancel{4}\sqrt{5}}=\frac{1}{\sqrt{5}}}$

(c) if at A, $\cos\theta=1/\sqrt{5}$, subbing this into our polar equation

$r=3+\cancel{\sqrt{5}}\left(\frac{1}{\cancel{\sqrt{5}}}\right)$

$=3+1$

$\boxed{\Rightarrow r=4}$

DO NOT WRITE IN THIS AREA

Question 4 continued

(Total for Question 4 is 7 marks)

5. The points representing the complex numbers $z_1 = 35 - 25\text{i}$ and $z_2 = -29 + 39\text{i}$ are opposite vertices of a regular hexagon, H, in the complex plane.

The centre of H represents the complex number α

(a) Show that $\alpha = 3 + 7\text{i}$

(2)

Given that $\beta = \dfrac{1+\text{i}}{64}$

(b) show that

$$\beta(z_1 - \alpha) = 1$$

(2)

The vertices of H are given by the roots of the equation

$$(\beta(z - \alpha))^6 = 1$$

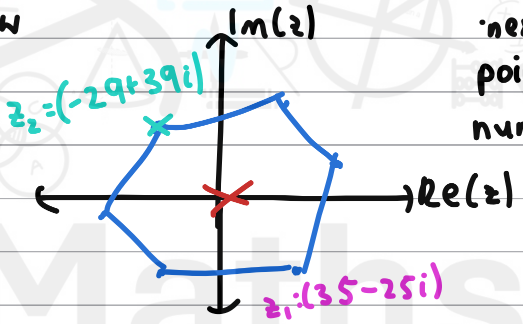
(c) (i) Write down the roots of the equation $w^6 = 1$ in the form $re^{\text{i}\theta}$

(1)

(ii) Hence, or otherwise, determine the position of the other four vertices of H, giving your answers as complex numbers in Cartesian form.

(4)

(a) complex plane hints at need to sketch the regular hexagon on an Argand diagram eg. as shown below



· next sketching on the two complex points : remembering that complex numbers of form a+bi is represented as (a,b) on plane

$z_1 = 35-25i$

$z_2 = -29+39i$

hence the centre of A (indicated) must be the midpoint of the two complex points

$\alpha = \left(\frac{x_1 - x_2}{2}, \frac{y_1 - y_2}{2}\right)$

$= \left(\frac{35+(-29)}{2} + \frac{-25+39}{2}i\right)$

$\therefore \alpha = \left(\frac{6}{2} + \frac{14}{2}i\right)$

$\Rightarrow \boxed{\alpha = 3+7i}$

(b) subbing β, z_1 and α into the L.H.S of the equation

$\frac{1+i}{64}\left((35-25i)-(3+7i)\right)$

P 7 2 7 9 5 A 0 1 4 3 2

Question 5 continued

evaluating brackets

$\frac{1+i}{64}(35-3+i(-25-7))$

$\Rightarrow \frac{1+i}{64}(32-32i)$ OR take $\frac{1}{64}$ out :

$\times 64$ $\times 64$

$\Rightarrow (1+i)(\frac{1}{2}-\frac{1}{2}i)$

expand brackets

$= \frac{1}{2} - \cancel{\frac{1}{2}i} + \cancel{\frac{1}{2}i} - \frac{1}{2}(i^2)$

using $i^2=-1$

$= \frac{1}{2}+\frac{1}{2} = \boxed{1}$

$\frac{1}{64}[(1+i)(32-32i)]$

expand inside square brackets

$\frac{1}{64}[32-\cancel{32i}+\cancel{32i}-32(i^2)]$

using $i^2=-1$

$\frac{1}{64}[32+32]$

$= \frac{1}{64}(64) = \boxed{1}$

(c)(i) remembering steps to find roots of complex numbers

1. write down 1 in exponential form : e^{i0}

2. take 6th root of RHS (power as a FRACTION) - to get w_1

$w_1 = (e^{i0})^{1/6}$

$= e^{i0} = 1$

3. then remembering that e^{i0} is 2π-periodic i.e keep multiplying each root by $w = e^{2\pi/n}$ (nth root of unity)

$\therefore w = e^{2\pi/6} \Rightarrow e^{\pi/3}$

$\therefore w_1 = 1$

$w_2 = e^{\pi/3 i}$

$w_3 = e^{\pi/3 i} \times e^{\pi/3} = e^{2\pi/3 i}$

$w_4 = e^{2\pi/3 i} \times e^{\pi/3} = e^{\pi i} = -1$

$w_5 = e^{\pi i} \times e^{\pi/3} = e^{4\pi/3 i}$ (or in 'principal arg. form' $\frac{4\pi}{3} - 2\pi = -\frac{2}{3}\pi i$) $\therefore e^{-2\pi/3 i}$

$w_6 = e^{-2\pi/3 i} \times e^{\pi/3} = e^{5\pi/3 i}$ (or in 'principal arg. form' $\frac{5\pi}{3} - 2\pi = -\pi/3$) $\therefore e^{-\pi/3 i}$

$\boxed{\therefore 1, e^{\pi/3 i}, e^{2\pi/3 i}, e^{\pi i}, e^{-2/3 \pi i}, e^{-\pi/3 i}}$

(general form: $e^{i\frac{\pi k}{3}}$, for $k = 0,1,2,3,4,5$)

(ii) trying to apply answers to part (c)(i) to those in (c)(ii)

↳ especially by spotting difference between

$w^6 = 1$

and $(\beta(z-\alpha))^6 = 1$

$\Rightarrow w = \beta(z-\alpha)$

Question 5 continued

these are the 6 roots from (c)(i) of which general form is $e^{i\pi k/3}$

need to solve for 'z'

subbing in what we know:

$$\frac{1+i}{64}(z-3-7i) = e^{i\pi k/3}$$

$\times 64$ $\times 64$

$$\Rightarrow (1+i)(z-3-7i) = 64e^{i\frac{\pi k}{3}}$$

$\div 1+i$ $\div 1+i$

$$\Rightarrow z-3-7i = \frac{\overbrace{64e^{i\frac{\pi k}{3}}}^{\text{exponential form}}}{\underbrace{1+i}_{a+bi \text{ form}}}$$

rationalise, keeping the a+bi 'cartesian' form question is asking for

$$64e^{i\frac{\pi k}{3}} = 64\left(\cos\left(\frac{\pi k}{3}\right) + i\sin\left(\frac{\pi k}{3}\right)\right)$$

$$z-3-7i = \frac{\left(64\left(\cos\left(\frac{\pi k}{3}\right)+i\sin\left(\frac{\pi k}{3}\right)\right)\right)}{(1+i)} \quad \begin{matrix}\times(1-i)\\ \times(1-i)\end{matrix}$$

...numerator:

$$64\left(\cos\left(\frac{\pi k}{3}\right)+i\sin\left(\frac{\pi k}{3}\right)\right)(1-i)$$

real:

$$64\cos\left(\frac{\pi k}{3}\right) - i^2\sin\left(\frac{\pi k}{3}\right)$$

$$= 64\cos\left(\frac{\pi k}{3}\right) + 64\sin\left(\frac{\pi k}{3}\right)$$

imaginary:

$$64\sin\left(\frac{\pi k}{3}\right) - 64i\cos\left(\frac{\pi k}{3}\right)$$

$\therefore$ in 'a+bi' form: $64\cos\left(\frac{\pi k}{3}\right) + \sin\left(\frac{\pi k}{3}\right) + i\left(64\sin\frac{\pi k}{3} - 64\cos\left(\frac{\pi k}{3}\right)\right)$

... denominator:

using fact that $zz^* = \sqrt{a^2+b^2}$

$1^2+1^2=2$

$$\therefore z-3-7i = \frac{\left(\cancel{64}^{32}\cos\left(\frac{\pi k}{3}\right) + \cancel{64}^{32}\sin\left(\frac{\pi k}{3}\right)\right) + i\left(\cancel{64}^{32}\sin\frac{\pi k}{3} - \cancel{64}^{32}\cos\left(\frac{\pi k}{3}\right)\right)}{\cancel{2}\ 1}$$

cancelling and adding 3 to real and 7 to imaginary components

$$z_k = \left(32\cos\left(\frac{\pi k}{3}\right) + 32\sin\left(\frac{\pi k}{3}\right) + 3\right) + i\left(32\sin\left(\frac{\pi k}{3}\right) - 32\cos\left(\frac{\pi k}{3}\right) + 7\right)$$

now sub in k values from 0 to 5

$$z_0 = \left(32(\cos(0) + 32\sin(0)) + 3\right) + i\left(32\sin(0) - 32\cos(0) + 7\right)$$

$$= (32+3) + i(-32+7) = \text{[the given]}\ 35-25i$$

z_1: $\left(32\cos(\pi/3) + 32\sin(\pi/3) + 3\right) + i\left(32\sin(\pi/3) - 32\cos(\pi/3) + 7\right)$

$\cos(\pi/3) = 1/2$; $\sin(\pi/3) = \sqrt{3}/2$

$= (16 + 16\sqrt{3} + 3) + i(16\sqrt{3} - 16 + 7)$

$\boxed{= 19 + 16\sqrt{3} + i(-9 + 16\sqrt{3})}$

z_2: $\left(32\cos\left(\frac{2\pi}{3}\right) + 32\sin\left(\frac{2\pi}{3}\right) + 3\right) + i\left(32\sin\left(\frac{2\pi}{3}\right) - 32\cos\left(\frac{2\pi}{3}\right) + 7\right)$

$\cos\left(\frac{2\pi}{3}\right) = -1/2$ $\sin\left(\frac{2\pi}{3}\right) = \sqrt{3}/2$

$\boxed{= (-13 + 16\sqrt{3}) + i(23 + 16\sqrt{3})}$

z_3: $\left(32\cos(\pi) + 32\sin(\pi) + 3\right) + i\left(32\sin(\pi) - 32\cos(\pi) + 7\right)$

$\cos\pi = -1$ $\sin\pi = 0$

$= (-32 + 0 + 3) + i(32 + 7)$

$= -29 + 39i$

z_4: $\left(32\cos\left(\frac{4\pi}{3}\right) + 32\sin\left(\frac{4\pi}{3}\right) + 3\right) + i\left(32\sin\left(\frac{4\pi}{3}\right) - 32\cos\left(\frac{4\pi}{3}\right) + 7\right)$

$\cos\left(\frac{4\pi}{3}\right) = -1/2$ $\sin\left(\frac{4\pi}{3}\right) = -\sqrt{3}/2$

$(-16 - 16\sqrt{3} + 3) + i(-16\sqrt{3} + 16 + 7)$

$\boxed{= (-13 - 16\sqrt{3}) + i(-16\sqrt{3} + 23)}$

z_5: $\left(32\cos\left(\frac{5\pi}{3}\right) + 32\sin\left(\frac{5\pi}{3}\right) + 3\right) + i\left(32\sin\left(\frac{5\pi}{3}\right) - 32\cos\left(\frac{5\pi}{3}\right) + 7\right)$

$\cos\left(\frac{5\pi}{3}\right) = 1/2$ $\sin\left(\frac{5\pi}{3}\right) = -\sqrt{3}/2$

$(16 - 16\sqrt{3} + 3) + i(-16\sqrt{3} - 16 + 7)$

$\boxed{= (19 - 16\sqrt{3}) + i(-16\sqrt{3} - 9)}$

$\therefore$ the 4 other vertices are:

$$\boxed{\begin{array}{l} 19 + 6\sqrt{3} + i(-9 + 16\sqrt{3});\ (16\sqrt{3} - 13) + i(23 + 16\sqrt{3});\ (-13 - 16\sqrt{3}) + i(23 - 16\sqrt{3}); \\ (19 - 16\sqrt{3}) - i(16\sqrt{3} + 9) \end{array}}$$

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Question 5 continued

(Total for Question 5 is 9 marks)

6. Given that

$$y = e^{2x} \sinh x$$

prove by induction that for $n \in \mathbb{N}$

$$\frac{d^n y}{dx^n} = e^{2x}\left(\frac{3^n + 1}{2}\sinh x + \frac{3^n - 1}{2}\cosh x\right)$$

(6)

proving by induction i.e proving a conjecture is true for all $n \in \mathbb{N}$

Step 1: base case

prove true for $n=1$;

LHS

$$\frac{d^1 y}{dx^1} = \frac{dy}{dx}$$

differentiate y wrt x using PRODUCT RULE

$$y = e^{2x}\sinh x$$

$u = e^{2x}$ $\quad v = \sinh x$ using $\frac{d}{dt}(\sinh x) = \cosh x$

$u' = 2e^{2x}$ $\quad v' = \cosh x$

$\therefore$ using $\frac{d}{dx}(uv) = u'v + uv'$

$$\frac{dy}{dx} = e^{2x}\cosh x + 2e^{2x}\sinh x$$

[NOTE: could've also done it in exponential form using exponential definitions of cosh x and sinh x but notice how $\frac{d^n y}{dx^n}$ expression doesn't involve exponentials $\therefore$ just use $\frac{d}{dx}(\sinh x) = \cosh x$]

RHS:

$$e^{2x}\left(\frac{3^1+1}{2}\sinh x + \frac{3^1-1}{2}\cosh x\right)$$

$$= e^{2x}(2\sinh x + \cosh x)$$

LHS = RHS $\therefore$ true for $n=1$

Step 2: assumption step

assume true for $n=k$

$$\frac{d^k y}{dx^k} = e^{2x}\left(\frac{3^k+1}{2}\sinh x + \frac{3^k-1}{2}\cosh x\right)$$

step 3: induction step

prove true for $n=k+1$

LHS :

$$\frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$$

$$= \frac{d}{dx}\left(e^{2x}\left(\frac{3^k+1}{2}\sinh x + \frac{3^k-1}{2}\cosh x\right)\right)$$

using product rule

$$2e^{2x}\left(\frac{3^k+1}{2}\sinh x + \frac{3^k-1}{2}\cosh x\right) + e^{2x}\left(\frac{3^k+1}{2}\cosh x + \frac{3^k-1}{2}\sinh x\right)$$

AIM:

$$\frac{d^{k+1}y}{dx^{k+1}} = e^{2x}\left(\frac{3^{k+1}+1}{2}\sinh x + \frac{3^{k+1}-1}{2}\cosh x\right)$$

looking at AIM : factorise e^{2x} out

$$e^{2x}\left(\cancel{2}\left(\frac{3^k+1}{\cancel{2}}\sinh x + \frac{3^k-1}{\cancel{2}}\cosh x\right) + \frac{3^k+1}{2}\cosh x + \frac{3^k-1}{2}\sinh x\right)$$

group $\sinh x$ and $\cosh x$

$$e^{2x}\left(\sinh x\left(3^k+1 + \frac{3^k-1}{2}\right) + \cosh x\left(3^k-1+\frac{3^k+1}{2}\right)\right)$$

common denominator

$$e^{2x}\left(\sinh x\left(\frac{2(3^k+1)+3^k-1}{2}\right) + \cosh x\left(\frac{2(3^k-1)}{2}+3^k+1\right)\right)$$

expand brackets

$$e^{2x}\left(\sinh x\left(\frac{2(3^k)+2+3^k-1}{2}\right) + \cosh x\left(\frac{2(3^k)-2+3^k+1}{2}\right)\right)$$

collect like terms

$$e^{2x}\left(\sinh x\left(\frac{3(3^k)+1}{2}\right) + \cosh x\left(\frac{3(3^k)-1}{2}\right)\right)$$

then using index power laws $\Rightarrow 3^1 \times 3^k = 3^{k+1}$

$$e^{x}\left(\sinh x\left(\frac{3^{k+1}+1}{2}\right) + \cosh x\left(\frac{3^{k+1}-1}{2}\right)\right) = \text{AIM}$$

$\therefore$ true for $n=k+1$

step 4: conclusion

since true for $n=1$, if true for $n=k$ and true for $n=k+1$, then true for all $n\in\mathbb{N}$

Question 6 continued

(Total for Question 6 is 6 marks)

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

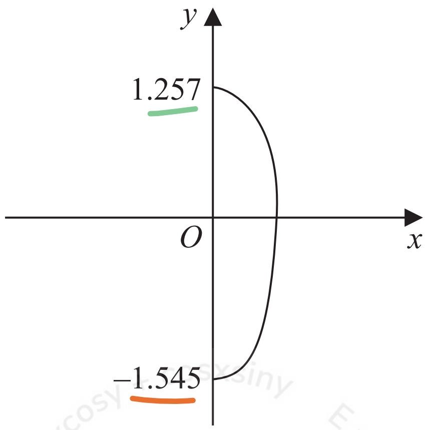

Figure 2

John picked 100 berries from a plant.

The largest berry picked was approximately 2.8 cm long.

$(1.257-(-1.545)$
$= 1.257+1.545$
$= 2.802)$

The shape of this berry is modelled by rotating the curve with equation

$$16x^2 + 3y^2 - y\cos\left(\frac{5}{2}y\right) = 6 \qquad x \geqslant 0$$

shown in Figure 2, about the y-axis through 2π radians, where the units are cm.

Given that the y intercepts of the curve are –1.545 and 1.257 to four significant figures,

(a) use algebraic integration to determine, according to the model, the volume of this berry.

(6)

Given that the 100 berries John picked were then squeezed for juice,

(b) use your answer to part (a) to decide whether, in reality, there is likely to be enough juice to fill a 200 cm^3 cup, giving a reason for your answer.

(2)

(a) remembering the formula for volumes of revolution about the y-axis

$V = \pi\int_{\alpha}^{\beta} x^2\,dy$

now need to substitute in information from the question – $\alpha = -1.545$
$\beta = 1.257$

here need to use these limits
as can't exploit any symmetry!

and need 'x^2' from given equation

Question 7 continued

$16x^2 + 3y^2 - y\cos(5/2\, y) = 6$

$16x^2 = 6 - 3y^2 + y\cos(5/2\, y)$

$\div 16 \qquad \div 16$

$x^2 = \frac{1}{16}(6 - 3y^2 + y\cos(5/2\, y))$

subbing into formula:

$V = \frac{1}{16}\pi \int_{-1.545}^{1.257} (6 - 3y^2 + y\cos(5/2\, y))\, dy$

integrating $- \int y\cos(5/2\, y)\, dy$ BY PARTS

let $u = y \quad v' = \cos(5/2\, y)$ } using $\int \cos kt\, dt = \frac{1}{k}\sin kt + c$

$u' = 1 \quad v = 2/5 \sin(5/2\, y)$

$\int y\cos(5/2\, y)\, dy = 2/5\, y\sin(5/2\, y) - 2/5 \int \sin(5/2\, y)\, dy$

$= 2/5\, y\sin(5/2\, y) + \frac{2}{5}\left(\frac{2}{5}\right)\cos(5/2\, y) + c$

$= 2/5\, y\sin(5/2\, y) + \frac{4}{25}\cos(5/2\, y) + c$

$V = \frac{1}{16}\pi\left[6y - y^3 + \frac{2}{5}y\sin\left(\frac{5}{2}y\right) + \frac{4}{25}\cos(5/2\, y)\right]_{-1.545}^{1.257}$

$= \frac{1}{16}\pi\Big\{\Big[6(1.257) - (1.257)^3 + \frac{2}{5}(1.275)(\sin(5/2 \times 1.257) + \frac{4}{25}\cos\left(\frac{5}{2}(1.257)\right)\Big]$

$- \Big[6(-1.545) - (-1.545)^3 + \frac{2}{5}(-1.545)\sin\left(\frac{5}{2} \times -1.545\right) + \frac{4}{25}\cos\left(\frac{5}{2}(-1.545)\right)\Big]$

evaluate this on CALC - hint : better to in calc press 1.257 then ANS for substitution and same for -1.545

$\frac{1}{16}\pi(5.3954\ldots - (-6.110\ldots))$

$= \frac{1}{16}\pi(11.50558\ldots)$

$= 2.25911\ldots$

$\boxed{= 2.26\,\text{cm}^3 \text{ (3 s.f)}}$

(b) max volume for 100 berries

$= 100 \times 2.2591\ldots$

$= 225.91\ldots$

$= 226\,\text{cm}^3$

but realistically not all the berry will become juice (eg. skin, pulp, seeds) $\therefore$ berries not likely to produce $200\,\text{cm}^3$ of juice

Question 7 continued

OR (also accepted):

200ml is less than 90% of possible 226ml ∴ may fill cup with juice

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Question 7 continued

(Total for Question 7 is 8 marks)

8. Given that a cubic equation has three distinct roots that all lie on the same straight line in the complex plane,

(a) describe the possible lines the roots can lie on.
(2)

$$f(z) = 8z^3 + bz^2 + cz + d$$

where b, c and d are real constants.

The roots of $f(z)$ are distinct and lie on a straight line in the complex plane.

Given that one of the roots is $\frac{3}{2} + \frac{3}{2}i$

(b) state the other two roots of $f(z)$
(1)

$$g(z) = z^3 + Pz^2 + Qz + 12$$

where P and Q are real constants, has 3 distinct roots.

The roots of $g(z)$ lie on a different straight line in the complex plane than the roots of $f(z)$

Given that

- $f(z)$ and $g(z)$ have one root in common
- one of the roots of $g(z)$ is -4

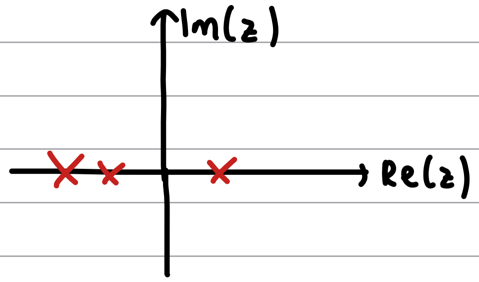
(c) (i) write down the value of the common root,
(1)

(ii) determine the value of the other root of $g(z)$
(3)

(d) Hence solve the equation $f(z) = g(z)$
(4)

(a) considering the two possible cases for the cubic equation to have 3 distinct roots (according to Fundamental Theorem of Algebra)

CASE 1: the cubic has three real roots



$\therefore$ must lie along the real axis $(y=0)$

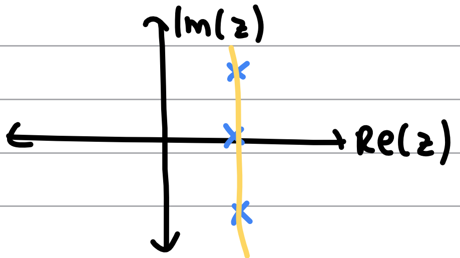
CASE 2: the cubic has one real root and one complex conjugate pair

P 7 2 7 9 5 A 0 2 4 3 2

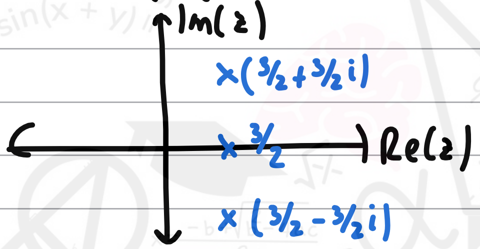
Question 8 continued

$\therefore$ for complex conjugate pair to lie on same line as the real solutions – their 'x' coordinate (ie real components) MUST be equal

$\therefore$ would lie on a vertical line ($x = \lambda$)



(b) continuing from part (a), f(z)'s roots must fit into CASE 2 i.e one real root and a complex conjugate pair that lie on the same vertical line



given $z = 3/2 + 3/2\, i$

$z^* = 3/2 - 3/2\, i$

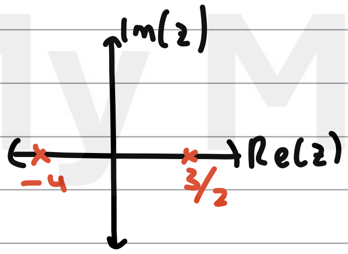
and all three roots' real components must be equal

$\therefore z_3 = 3/2$

$\boxed{\therefore 3/2,\ 3/2 - 3/2\, i,\ 3/2 + 3/2\, i}$

(c)(i) now we're dealing with another cubic equation $g(z) = z^3 + Pz^2 + Qz + 12$ already given one root $= -4$ but straight away can see $\frac{3}{2} \pm \frac{3}{2} i$ cannot be common to g(z) as $Re(z_1, z_2) \neq -4$

$\therefore$ wouldn't lie on same line



$\therefore$ common root $= 3/2$

(straight line = real axis $(y=0)$)

(ii) METHOD 1: factor theorem and by inspection

given that roots $g(z) = -4, 3/2$ then use fact that if $f(z)=0$, $(x - z)$ is a factor to form a cubic that can then expand and compare coefficients

$(z - 3/2)(z+4)(z+\alpha) = z^3 + Pz^2 + Qz + 12$

then compare CONSTANT TERMS

$(-3/2)(4)(\alpha) = 12$

$\Rightarrow -6\alpha = 12 \quad \div -6$

$\Rightarrow \boxed{\alpha = -2}$

METHOD 2: using roots of polynomials formulae

using given cubic equation

$$g(z) = z^3 + \underset{\Sigma\alpha}{P}z^2 + \underset{\Sigma\alpha\beta}{Q}z + \underset{\alpha\beta\gamma}{12}$$

where:

· sum of roots: $\Sigma\alpha = -b/a = -P$

· Sum of product pairs: $\Sigma\alpha\beta = c/a = Q$

· product of roots: $\alpha\beta\gamma = -d/a = -12$

let $\alpha = -4, \beta = 3/2, \gamma = \gamma$

$\therefore$ using 'product of roots'

$-4(3/2)(\gamma) = -12$

$\Rightarrow -6\gamma = -12$ $\quad \div -6 \quad \div -6$

$\Rightarrow \gamma = 2$ $\quad \boxed{\therefore \text{root} = 2}$

(d) question is to solve $f(z) = g(z)$ i.e to find p.o.i of the two cubics

WAY 1:

consider $f(z)$ roots:

$z = 3/2, \ 3/2 \pm 3/2 i$

$\therefore$ rewriting (factor theorem!)

$f(z) = (z - 3/2)(z - (3/2 + 3/2 i))(z - (3/2 - 3/2 i))$

$\times 2 \qquad \times 2 \qquad \times 2$

$= (2z-3)((2z-3)-3i)((2z-3)+3i)$

expanding two brackets

$= (2z-3)((2z-3)^2+9)$

expand 2nd to form quadratic

$= (2z-3)(4z^2-12z+9)+9)$

$= (2z-3)(4z^2-12z+18)$

consider $g(z)$ roots are $z = 3/2, -4, 2$

$g(z) = (z - 3/2)(z+4)(z-2)$

expand 2 last brackets

$g(z) = (z - 3/2)(z^2+2z-8)$

$\therefore$ Subbing in what want to solve

$(2z-3)(4z^2-12z+18) = (z-3/2)(z+4)(z-2)$

$(2z-3)(4z^2-12z+18) = (z-3/2)(z^2+2z-8)$

trying to get to cancel $\therefore$ take 2 out of LHS

$2(z-3/2)(4z^2-12z+18) = (z-3/2)(z^2+2z-8)$

$\times 2$

$(z-3/2)(8z^2-24z+36) - (z-3/2)(z^2+2z-8) = 0$

factorise $(z-3/2)$ out

$(z-3/2)\left[(8z^2-24z+36)-(z^2+2z-8)\right] = 0$

$(z-3/2)(7z^2-26z+44) = 0$

↳ quadratic formula

$z = \dfrac{26 \pm \sqrt{(-26)^2-4(44)(7)}}{14}$

$= \dfrac{26 \pm \sqrt{-556}}{14}$

eval. $\sqrt{-556}$ using $\sqrt{-1} = i$

$= \dfrac{26 \pm i2\sqrt{139}}{14}$

$\div 2$ $\div 2$

$= \dfrac{13 \pm i\sqrt{139}}{7}$

$\boxed{\Rightarrow z = 3/2,\ \dfrac{13 \pm i\sqrt{139}}{7}}$

WAY 2: using 'scalar multiples' and equating straight away

consider $f(z) \to$ roots $z = 3/2,\ 3 \pm 3/2\, i$

$\therefore$ rewriting (factor theorem)

$f(z) = (z-3/2)(z-(3/2+3/2\, i))(z-(3/2-3/2\, i))$

$g(x) \to$ roots: $z = 3/2, -4, 2$

$\therefore$ rewriting (factor theorem)

$g(x) = (z-3/2)(z+4)(z-2)$

EQUATING $f(z) = g(z)$

$(z-3/2)(z-(3/2+3/2\, i))(z-(3/2-3/2\, i)) = (z-3/2)(z+4)(z-2)$

but realising that on L.H.S

$f(z) = 8z^3+bz^2+cz+d$ $\therefore$ need to scale by 8

Question 8 continued

$\cancel{8}(z-\cancel{3/2})(z-(3/2+3/2\,i))(z-(3/2-3/2\,i)) = (z-\cancel{3/2})(z+4)(z-2)$

expand brackets

$8z^2-24z+36 = z^2+2z-8$

$\Rightarrow 7z^2-26z+44=0$

quadratic formula

$$\frac{-(-26)\pm\sqrt{(-26)^2-4(7)(44)}}{14}$$

$$= \frac{26\pm\sqrt{-556}}{14} = \frac{\cancel{26}^{13}\pm\cancel{2}\sqrt{139}\,i}{\cancel{14}_{7}}$$

$$\boxed{=\frac{13\pm\sqrt{139}\,i}{7}}$$

AND common solution from (c)(i)

$\boxed{3/2}$

(Total for Question 8 is 11 marks)

9. A patient is treated by administering an antibiotic intravenously at a constant rate for some time.

Initially there is none of the antibiotic in the patient.

At time t minutes after treatment began

- the concentration of the antibiotic in the blood of the patient is x mg/ml
- the concentration of the antibiotic in the tissue of the patient is y mg/ml

The concentration of antibiotic in the patient is modelled by the equations

$$\frac{dx}{dt} = 0.025y - 0.045x + 2 \quad - ①$$

$$\frac{dy}{dt} = 0.032x - 0.025y \quad - ②$$

(a) Show that

$$40\,000\frac{d^2y}{dt^2} + 2800\frac{dy}{dt} + 13y = 2560$$

(3)

(b) Determine, according to the model, a general solution for the concentration of the antibiotic in the patient's tissue at time t minutes after treatment began.

(5)

(c) Hence determine a particular solution for the concentration of the antibiotic in the tissue at time t minutes after treatment began.

(4)

To be effective for the patient the concentration of antibiotic in the tissue must eventually reach a level between 185 mg/ml and 200 mg/ml.

(d) Determine whether the rate of administration of the antibiotic is effective for the patient, giving a reason for your answer.

(2)

(a) notice how the 'show that' 2 O.D.E is given in terms of 'y' and its derivatives → this implies that need to use the 2 differential equations to eliminate 'x'

rearrange ② for 'x'

$0.032x = \frac{dy}{dt} + 0.025y$

$\div 0.032 \qquad \div 0.032$

$\Rightarrow x = \frac{1}{0.032}\left(\frac{dy}{dt} + 0.025y\right)$

next differentiate for $\frac{dx}{dt}$

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Question 9 continued

$$\frac{dx}{dt} = \frac{1}{0.032}\left(\frac{d^2y}{dx^2} + 0.025\frac{dy}{dt}\right)$$

and sub x and $\frac{dx}{dt}$ into ① to eliminate x and its derivatives

$$\frac{1}{0.032}\left(\frac{d^2y}{dt^2} + 0.025\frac{dy}{dt}\right) = 0.025y - 0.045\left(\frac{1}{0.032}\left(\frac{dy}{dt} + 0.025y\right)\right) + 2$$

expand brackets

$$\frac{125}{4}\frac{d^2y}{dt^2} + \frac{25}{32}\frac{dy}{dt} = \frac{1}{40} - \frac{45}{32}\frac{dy}{dt} - \frac{9}{256}y + 2$$

collect like terms

$$\frac{125}{4}\frac{d^2y}{dt^2} + \frac{35}{16}\frac{dy}{dt} + \frac{13}{1280}y = 2$$

$\times 1280$ to get show that (notice how 'y' coefficient compares with that of given 'show that' equation

$$\boxed{\Rightarrow 40{,}000\frac{d^2y}{dt^2} + 2{,}800\frac{dy}{dt} + 13y = 2560}$$

(b) to get tissue - need to solve above non-homogenous 2 O.D.E

A.E : $40{,}000m^2 + 2{,}800m + 13 = 0$

equation solver on calc

$$m = \frac{-2800 \pm \sqrt{(2800)^2 - 4(40{,}000)(13)}}{80{,}000}$$

$$= \frac{-2800 \pm \sqrt{5{,}760{,}000}}{80{,}000}$$

$$= -\frac{1}{200} = -0.005$$

G.S remembering general solution to 2.O.D.Es with 2 real roots, α and β $(b^2 - 4ac) > 0$

$$y = Ae^{\alpha x} + Be^{\beta x}$$

C.F : $y = Ae^{-0.005t} + Be^{-0.065t}$

now P.I

try $y = \lambda$

$y' = 0$

$y'' = 0$

Form of $f(x)$	Form of particular integral
k	λ
$ax + b$	$\lambda + \mu x$
$ax^2 + bx + c$	$\lambda + \mu x + \nu x^2$
ke^{px}	λe^{px}
$m\cos\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$
$m\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$
$m\cos\omega x + n\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$

WARNING!
The particular integral must not contain any term in the complementary function. If it does, you'll need to add an x and possibly even an x^2 in front of your usual PI form

Question 9 continued

subbing into original 2 O.D.E:

$$40,000(0) + 2,800(0) + 13\lambda = 2560$$

$$\Rightarrow 13\lambda = 2560$$

$$\div 13 \qquad \div 13$$

$$\lambda = \frac{2560}{13}$$

$$\therefore y = Ae^{-0.05t} + Be^{-0.065t}$$

G.S = C.F + P.I $\boxed{y = Ae^{-t/20} + Be^{-13/200t} + \frac{2560}{13}}$

(c) using initial conditions at $t=0$, $y=0$

$$0 = Ae^{(-0)/200} + Be^{-13/200(0)} + \frac{2560}{13}$$

$$0 = A + B + \frac{2560}{13}$$

$$\Rightarrow A + B = -\frac{2560}{13} \quad -(1)$$

at $t=0$, $\frac{dy}{dt} = 0$

↳ using exponential differentiation

$$\frac{dy}{dt} = -0.005Ae^{-0.005t} - 0.065e^{-0.065t}$$

$$0 = -0.005(A) - 0.065B \quad -(2)$$

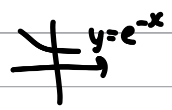
solving for 'A' and 'B' - eqtn solver

$$A = -\frac{640}{3}$$

$$B = \frac{640}{39}$$

$$\boxed{y = -\frac{640}{3}e^{-0.005t} + \frac{640}{39}e^{-0.065t} + \frac{2560}{13}}$$

(d) 'eventually' suggests evaluate 'y' at $t \to \infty$



as $t \to \infty$, $e^{-0.005t} \to 0$

$$e^{-0.065t} \to 0$$

$$\therefore y \to \frac{2560}{13} = 196.92...$$

$\therefore$ rate of administration sufficient to meet required level

Question 9 continued

(Total for Question 9 is 14 marks)

TOTAL FOR PAPER IS 75 MARKS